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A PROBLEM IN GEOMETRY CONNECTED WITH THE ANALYTIC CONTINUATION OF A POWER SERIES.*

By T. H. Gronwall.

Consider a power series $\sum_0^\infty c_n z^n$ convergent for $|z| < 1$, and such that the analytic function defined by it has the only singularity $z = -1$ at finite distance. At any point $z_0 \neq -1$ at finite distance, the function may then be expanded in powers of $z - z_0$, convergent for $|z - z_0| < |1 + z_0|$, and this power series is obtainable through analytic continuation of the original series successively to a number of points $a_1, a_2, \cdots, a_m, a_{m+1} = z_0$, each point lying within the circle of convergence of the power series belonging to the preceding point, so that

$$|a_1| < 1, \qquad |a_\nu - a_{\nu-1}| < |1 + a_{\nu-1}|, \qquad \nu = 1, 2, \cdots, m+1.$$

The question now arises: what is the *minimum* number m of intermediate points $a_1, a_2, \cdots, a_m$ necessary for the analytic continuation of the given power series to the point z_0?

It will be convenient to refer the points z to a polar coördinate system with the pole at $z = -1$ and the real axis as reference line, so that

$$z = -1 + re^{\theta i}, \qquad -\pi \leqq \theta \leqq \pi.$$

The set of points z_0 for which no intermediate points are necessary, or $m = 0$, evidently consists of the interior of the circle $|z| = 1$ or

$$r = 2 \cos \theta, \qquad -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}.$$

Generally, *the set of points z_0 for which no more than m intermediate points are necessary, consists of the interior of the curve*

$$(C_m) \qquad r = \left(2 \cos \frac{\theta}{m+1}\right)^{m+1}, \qquad -\pi \leqq \theta \leqq \pi.\dagger$$

This statement being true for $m = 0$, we shall assume it proved for $C_0, C_1, \cdots, C_{m-1}$ and show its validity for C_m. Any point $z_0 = -1 + re^{\theta i}$ for which m or fewer intermediate points are necessary, must be interior to a circle passing through $z = -1$, with center at some point $z' =$

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* Read before the American Mathematical Society, Sept. 2, 1916.

† For $m = 0$, we have instead $-(\pi/2) \leqq \theta \leqq (\pi/2)$.

$-1 + r'e^{\theta' i}$ inside C_{m-1}, so that

$$r < 2r' \cos(\theta' - \theta), \qquad -\frac{\pi}{2} < \theta' - \theta < \frac{\pi}{2},$$

and since $r' < [2 \cos(\theta'/m)]^m$, we obtain

$$(1) \qquad \begin{aligned} f(\theta') &= 2^{m+1}\left(\cos\frac{\theta'}{m}\right)^m \cos(\theta' - \theta) - r > 0, \\ & -\frac{\pi}{2} < \theta' - \theta < \frac{\pi}{2}, \qquad -\pi \leqq \theta,\ \theta' \leqq \pi. \end{aligned}$$

To obtain the upper bound of all r belonging to a given θ, we must evidently make $f(\theta')$ a maximum in respect to θ'; forming the derivative, we find the condition

$$(2) \qquad f'(\theta') = 2^{m+1}\left(\cos\frac{\theta'}{m}\right)^{m-1} \sin\left(\theta - \frac{m+1}{m}\theta'\right) = 0.$$

The first factor does not vanish for $-\pi \leqq \theta' \leqq \pi$ (except for $m = 2$, $\theta' = \pm\pi$, which gives $f(\theta') < 0$), so that all solutions of (2) are given by

$$\theta' = \frac{m}{m+1}(\theta - k\pi), \qquad k \text{ integer.}$$

The condition $-(\pi/2) < \theta' - \theta < (\pi/2)$ becomes

$$-\frac{\pi}{2} - \frac{mk}{m+1}\pi < \frac{\theta}{m+1} < \frac{\pi}{2} - \frac{mk}{m+1}\pi,$$

and the condition $-\pi \leqq \theta' \leqq \pi$ gives

$$\frac{k\pi}{m+1} - \frac{\pi}{m} \leqq \frac{\theta}{m+1} \leqq \frac{k\pi}{m+1} + \frac{\pi}{m}.$$

Combining these two sets of inequalities, we find

$$-\frac{\pi}{2} - \frac{mk}{m+1}\pi < \frac{k\pi}{m+1} + \frac{\pi}{m}, \qquad \frac{k\pi}{m+1} - \frac{\pi}{m} < \frac{\pi}{2} - \frac{mk\pi}{m+1},$$

or

$$-\frac{1}{m} - \frac{1}{2} < k < \frac{1}{m} + \frac{1}{2},$$

whence $k = 0$.* Consequently

$$(3) \qquad \theta' = \frac{m}{m+1}\theta,$$

and (1) gives $r < [2 \cos(\theta/m + 1)]^{m+1}$, which proves our proposition.

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* Except possibly for $m = 1$. But then the condition $-\pi \leqq \theta' \leqq \pi$ is replaced by $-(\pi/2) \leqq \theta' \leqq (\pi/2)$, and we obtain $-(1/2m) - \frac{1}{2} < k < (1/2m) + \frac{1}{2}$, so that $k = 0$.

The above calculation is visibly identical with the one leading to the equation of the envelope of the circles $r = 2r' \cos(\theta' - \theta)$ passing through $z = -1$ and having their centers on the curve C_{m-1} or $r' = [2\cos(\theta'/m)]^m$. The curve C_m is not, however, the envelope of all these circles, since on account of $-\pi \leqq \theta \leqq \pi$, (3) confines θ' to the interval

$$-\frac{m\pi}{m+1} \leqq \theta' \leqq \frac{m\pi}{m+1}.\dagger$$

The circles belonging to values of θ' in the remainder of the interval $-\pi \leqq \theta' \leqq \pi$ evidently have as envelope the two arcs of the complete curve

$$r = \left(2\cos\frac{\theta}{m+1}\right)^{m+1}, \qquad -\frac{m+1}{m}\pi \leqq \theta \leqq -\pi \quad \text{and} \quad \pi \leqq \theta \leqq \frac{m+1}{m}\pi.$$

The complete curve being symmetrical about the real axis, these two arcs are evidently also given by

$$(4)\quad \begin{aligned} r &= \left(2\cos\frac{2\pi-\theta}{m+1}\right)^{m+1}, &\quad \frac{m-1}{m}\pi \leqq \theta \leqq \pi,\\ r &= \left(2\cos\frac{2\pi+\theta}{m+1}\right)^{m+1}, &\quad -\pi \leqq \theta \leqq -\frac{m-1}{m}\pi. \end{aligned}$$

By C_m' we shall denote the closed curve formed by these two arcs and the arc of circle given by

$$(5)\quad r = 2^{m+1}\left(\cos\frac{\pi}{m}\right)^m \cos(\pi - \theta), \qquad \frac{m-1}{m}\pi \leqq \theta \leqq \pi \quad \text{and} \quad -\pi \leqq \theta \leqq -\frac{m-1}{m}\pi;$$

this circle evidently passes through $z = -1$ and has as center the double point $\theta = \pi$ of C_{m-1}.

The significance of C_m' for the analytic continuation will appear in the following problem: given a point $z_0 = -1 + re^{\theta i}$ inside C_m but out-

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† If we consider the *complete* curve $r = (2\cos\theta/m + 1)^{m+1}$, i. e., let θ vary through all the values for which this equation gives $r \geqq 0$, it is readily seen that each of its points is obtained once when θ varies through the interval $-(m+1)\pi/2 < \theta \leqq (m+1)\pi/2$. The complete curve has double points at $\theta = k\pi$, $k = 1, 2, \cdots, \left[\frac{m+1}{2}\right]$ and consists of $\left[\frac{m+1}{2}\right] + 1$ loops, each interior to the preceding one. This curve is the special case $n = 1/(m+1)$ of the sine spiral $r^n = a^n \cos n\theta$, for the general properties of which compare H. Wieleitner, Spezielle ebene Kurven (Leipzig, 1908), pp. 131–142, and G. Loria, Spezielle algebraische und transcendente ebene Kurven, 2d ed., vol. 1 (Leipzig, 1908), pp. 470–482.

side or on C_{m-1}, so that the analytic continuation to z_0 requires at least m intermediate points $a_1, a_2, \cdots, a_m$; what are the possible positions of the last intermediate point a_m?

It is obvious that, since a_m is the center of a circle through $z = -1$ and containing z_0 in its interior, a_m and z_0 must lie on the same side of the perpendicular bisector of the segment joining z_0 to -1, and on the other hand, a_m must be interior to C_{m-1}.

We shall now see that for $m > 1$, *the possible positions of* a_m *thus defined form a single continuum, when* z_0 *is exterior to or on* C_m', *but two separate continua, when* z_0 *is interior to* C_m'. To prove this, it is evidently sufficient to show that the perpendicular bisector mentioned above intersects C_{m-1} in two and four points respectively. The equation of the perpendicular bisector in the coördinates r', θ' is

$$2r' \cos(\theta' - \theta) = r,$$

and hence the intersection points with C_{m-1} are given by

$$f(\theta') = 0, \qquad -\pi \leqq \theta' \leqq \pi,$$

where $f(\theta')$ is the function defined by (1). On account of the symmetry about the real axis, we may assume $0 \leqq \theta \leqq \pi$, and since $f(\theta') < 0$ unless $\cos(\theta' - \theta) > 0$, it follows from $-\pi \leqq \theta' \leqq \pi$ that any root of $f(\theta') = 0$ must lie in either of the intervals

$$-\frac{\pi}{2} < \theta' - \theta < \frac{\pi}{2} \qquad \text{or} \qquad -2\pi < \theta' - \theta < -\frac{3\pi}{2}.$$

Now let us examine the roots of $f'(\theta') = 0$ in these intervals. In the first, we have already found the only root (3); in the second we find, by substituting $\theta' = m(\theta - k\pi)/m + 1$ in the inequality defining the interval,

$$\frac{3\pi}{2} < \frac{\theta}{m+1} + \frac{mk}{m+1}\pi < 2\pi,$$

and combining this with $0 \leqq \theta \leqq \pi$, we obtain

$$\frac{3}{2} < \frac{mk+1}{m+1}, \qquad \frac{mk}{m+1} < 2,$$

whence $k = 2$ (and $m > 1$). Thus $f'(\theta') = 0$ has one root $\theta_1' = \dfrac{m(\theta - 2\pi)}{(m+1)}$ in the second, and another $\theta_2' = m\theta/(m+1)$ in the first of these intervals, and consequently either contains at most two roots of $f(\theta') = 0$. Now

$$f(\pm\pi) = -2^{m+1}\left(\cos\frac{\pi}{m}\right)^m \cos\theta - r,$$

$$
\begin{aligned}
f(\theta_1') &= 2^{m+1}\left(\cos\frac{2\pi-\theta}{m+1}\right)^{m+1} - r,\\
f(0) &= 2^{m+1}\cos\theta - r,\\
f(\theta_2') &= \left(2\cos\frac{\theta}{m+1}\right)^{m+1} - r,
\end{aligned}
\tag{6}
$$

and since z_0 is inside C_m, we have $f(\theta_2') > 0$; observing furthermore that $f(-\pi) = f(\pi)$ and that $f(0) \geqq 0$ implies $f(\pm\pi) < 0$, we find the only possible sign combinations to be the following:

$f(-\pi)$	$f(\theta_1')$	$f(0)$	$f(\theta_2')$	$f(\pi)$
+	+	−	+	+
+	−	−	+	+
−	+	+	+	−
−	+	−	+	−
−	−	+	+	−
−	−	−	+	−

so that $f(\theta') = 0$ has either two or four roots, the latter being the case when $f(\pm\pi) < 0$, $f(\theta_1') > 0$, $f(0) < 0$. Here $f(\pm\pi) < 0$ signifies that z_0 is outside the circle (5). The conditions $\theta \leqq \pi$ and $-\pi \leqq \theta_1'$ give $(m-1)\pi/m \leqq \theta \leqq \pi$, and consequently $f(\theta_1') > 0$ signifies that z_0 lies between the upper of the two arcs defined by (4) and the real axis. Hence z_0 is inside C_m', and since $\theta \geqq \pi/2$, we have also $f(0) < 0$, so that all the conditions for four roots are satisfied.